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# WAVE PROPAGATION IN TUBULAR BONES

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**Abstract**—By employing an analytical procedure, the dispersion relation for axisymmetric acoustic waves propagating along the axis of a long composite bone has been derived in the paper by considering energy dissiptation and bone piezoelectricity (as per previous experimental observations). The particular case, in which the piezoelectric effects are disregarded, is dealt with in greater detail. Numerical results for compact bones, computed on the basis of this particularization are also presented and thereby an attempt is made to illustrate the influence of dissipative material behaviour of bones on the wave propagation characteristics. The paper concludes with a discussion on the possible effects of bone piezoelectricity on the basis of plane wave approximation.

## 1. INTRODUCTION

Studies on the waves propagating through continuous media have the potential to provide useful techniques for the purpose of ascertaining the material characteristics when theoretical results are correlated with the corresponding experimental findings. For osseous media such a study offers further information regarding the pathological state, the site of fracture etc., in addition to those related to their mechanical and electrical properties.

It is now an established fact that bone is a two-phase, fibre re-inforced composite material. One phase is represented by the visco-elastic bonding and the second by the osteons. As early as 1957 Fukada and Yasuda ascertained the piezoelectricity of bones, while McElhancy[1] basing upon his experimental observations pointed out that bone exhibits viscoelastic properties. Sedlin[2] studied the viscoelastic effects of bone material by treating it as a standard linear solid, while in a poroelastic study of bone media, Nowinski[3] confirmed this assumption analytically. Through a wave propagation study, Lang[4] determined experimentally the anisotropic elastic moduli of bone material, while Yoon and Katz[5] established theoretically the hexagonal characteristics of bones. The viscoelastic properties of wet cortical bones were determined by Lakes *et al.*[6]. Gottesman and Hashin[7] analysed the viscoelastic behaviour of bones on the basis of their microstructure. Although in reality the microstructural composition of bones seems to be inhomogeneous, a gross bone can be approximately regarded as a homogeneous continuum. In fact, the present athors[8], in a separate communication, reported that the wave propagation characteristics of bone media are not seriously affected due to inhomogeneity. The problem of torsional wave propagation in tubular bones by accounting for their dissipative and piezoelectric effects was the subject of discussion in that paper.

The problem of wave propagation in a bone medium treated as a hexagonal material was studied by Vayo and Ghista[9]. But they did not account for the material damping and the piezoelectric effects of bone tissues. The purpose of the present study is to examine the effects of these material properties on the wave propagation characteristics. The analysis presented here is suitable for illustrating the two-phase behaviour of the osseous tissues described above. The study corresponds to a situation in which both the endosteal and periosteal surfaces of the tubular bone under consideration are maintained at zero potentials and free of tractions. While the dispersion relation is derived for the general case, the numerical computational work is carried out for the particular case when the bone piezoelectricity is not taken into account. It has of course been concluded that the piezoelectric properties may not have an appreciable effect on the mode of propagation of waves in an osseous medium, at least under the purview of a plane wave approximation. Further, due to non-availability of requisite experimental data, the computational work is based on the consideration of a compact bone.

## 2. ELASTO-DYNAMICS OF THE AXI-SYMMETRIC WAVES IN A PIEZOELECTRIC MEDIUM POSSESSING DAMPING MATERIAL BEHAVIOUR

As already mentioned, Fukada and Yasuda[10] seem to be first to illustrate the piezoelectricity in bone tissues. They carried out experiments by taking different specimens of dry bones. During the last two decades, the roles of piezoelectricity in remodelling of bones, fracture healing, and recovery from different bone diseases, have been the subject of quite a few investigations. Saha and Lakes[11] remarked that an attempt towards correlating the experimental findings of the wave propagation velocities and amplitude attenuations with the conclusions based on analytical studies is quite useful, especially when one wants to confirm a particular pathological state.

In this section, we present the solution of the elasto-dynamic equations of the axi-symmetric waves, based on the hexagonal-polar constitutive relations (see [12]). Let us consider an axial harmonic wave propagating along the longitudinal axis of the bone modelled as a two-layered cylindrical shell. The longitudinal axis is supposed to correspond to the material axis of hexagonal polar symmetry. With the axis of $z$ along the longitudinal axis of the tubular bone, let $(r, \theta, z)$ be the coordinates of a representative material point of the specimen. Due to the propagation of the axi-symmetric wave, the motion of the constituent particles undergo vibrations symmetric with respect to the $z$-axis, and consequently any wave field parameter (e.g. particle displacement, stress, strain, electric fields, electric displacement, electric potential) which is independent of $\theta$ can be represented by a space-time dependent function, $f(r, z, t)$ given by

$$f(r, z, t) = f(r) \exp [i(wt - kz)] \tag{1}$$

in which $w$ is the circular frequency and $k$ the wave propagation constant.

Now the constitutive relations for the stress $(T_m)$, strain $(S_n)$, electric field $(E_l)$ and the electric displacement $(D_k)$ for a piezoelectric medium, in general, are given by (see[13]):

$$T_m = C_{mn}S_n - e_{km}D_k \tag{2}$$

$$D_k = e_{kn}S_n - \epsilon_{lk}E \tag{3}$$

in which

$$1 \leq k, l \leq 3, 1 \leq m, n \leq 6$$

and the elastic moduli matrix $C_{mn}$, the piezoelectric matrix $e_{km}$ and dielectric matrix for bone are given in Güzelsu[12]. In the cylindrical polar co-ordinate $(r, \theta, z)$ system, $T_m$ stands for the stress components $T_{rr}$, $T_{\theta\theta}$, $T_{zz}$, $T_{\theta z}$, $T_{rz}$ and $T_{r\theta}$, while $S_n$ for the strain components $S_{rr}$, $S_{\theta\theta}$, $S_{zz}$, $S_{\theta z}$, $S_{rz}$ and $S_{r\theta}$ with $m = 1, 2, 3, 4, 5$ and 6 respectively. The displacement components along $r$-, $\theta$- and $z$-directions are assumed to be $u_r$, $u_\theta$ and $u_z$ respectively. The strain displacement relations as well as the equilibrium equations governing the motion of the cylindrical medium are given in [14]. In general, the elements in the piezo-electric matrix and the dielectric matrix like the elastic moduli in a viscoelastic medium are frequency-dependent[15]. But throughout in this analysis $e_{km}$ and $\epsilon_{lk}$ are considered as frequency-independent, whereas the frequency-dependence of $C_{mn}$ would be considered in the following section. It may, however, be mentioned in this connection that the elastic moduli are almost independent of frequency in the ultrasonic range[6].

The electric field as also the electric displacement induced by an acoustic wave propagating through the bone medium must satisfy the Maxwell equations of electrodynamics, as well as the constitutive relations (2) and (3). Since the acoustic waves are much slower than the electromagnetic waves, one may consider induced magnetic field quantities to be zero[13]. Hence according to this quasi-static approximation,

$$\nabla \cdot \mathbf{D} = 0 \tag{4}$$

and

$$\nabla \times \mathbf{E} = 0. \tag{5}$$

This last equation asserts the existence of a potential function $v$, such that

$$\mathbf{E} = -\nabla v. \tag{6}$$

In cylindrical polar co-ordinates, the eqn (6) reads

$$E_r = -\frac{\partial v}{\partial r}, E_\theta = -\frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{and} \quad E_z = -\frac{\partial v}{\partial z}. \tag{7}$$

As a wave field parameter, $v$ also satisfies equation (1) and consequently

$$E_\theta = 0. \tag{7a}$$

Now using the constitutive relations (2) and (3) in the equations of motion and Gauss' divergence equation (7) together with the other relevant relations we obtain the following set of partial differential equations

$$(a\nabla^2 + A)\nabla^2\varphi + B\nabla^2 u_z + C\nabla^2 v = 0 \tag{8}$$

$$(h\nabla^2 + A)\nabla^2\psi + e\nabla^2 v = 0 \tag{9}$$

$$B\nabla^2\varphi + (b\nabla^2 + D)u_z + (d\nabla^2 + F)v = 0 \tag{10}$$

$$C\nabla^2\varphi + e\nabla^2\psi + (d\nabla^2 + F)u_z + (G + f\nabla^2)v = 0 \tag{11}$$

with

$$\begin{aligned} &u_r = \frac{\partial\varphi}{\partial r}, u_\theta = \frac{\partial\psi}{\partial r}, \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \\ &a = C_{11}, A = \rho w^2 - k^2 C_{44}, B = -ik(C_{13} + C_{44}) \\ &C = -ik(e_{31} + e_{15}), b = C_{44}, D = \rho w^2 - k^2 C_{33}, d = e_{15} \\ &F = -k^2 e_{33}, e = -ike_{14}, f = -\epsilon_{11}, G = k^2\epsilon_{33}, h = C_{66} \end{aligned} \tag{12}$$

and $\varphi$, $\psi$, $u_z$ and $v$ are the $r$-dependent parts of the corresponding wave field parameters satisfying eqn (1). From the system of eqns (9)–(11) we obtain the following differential equation through a process of elimination

$$(\nabla^8 + a_0\nabla^6 + b_0\nabla^4 + c_0\nabla^2 + d_0)(u_z, \varphi, \psi, v) = 0 \tag{13}$$

i.e.

$$(\nabla^2 + \lambda_1^2)(\nabla^2 + \lambda_2^2)(\nabla^2 + \lambda_3^2)(\nabla^2 + \lambda_4^2)(u_z, \varphi, \psi, v) = 0 \tag{13a}$$

in which $-\lambda_i^2 (i = 1, 2, 3, 4)$ are the roots of the biquadratic equation

$$x^4 + a_0 x^3 + b_0 x^2 + c_0 x + d_0 = 0 \tag{13b}$$

and

$$a_0 = \frac{RM + SL + rm + sl}{RL + rl}, \quad b_0 = \frac{RN + SM + TL + rn + sm + tl}{RL + rl}$$
$$c_0 = \frac{SN + TM + sn + tm}{RL + rl}, \quad d_0 = \frac{NT + nt}{RL + rl},$$
$$L = ad, M = aF + Ad - BC, N = AF,$$
$$l = ab, m = Ab + aD - B^2, n = AD,$$
$$R = h(Bd - bC), S = h(BF - CD) + A(Bd - bC), T = A(BF - CD)$$
$$r = h(Cd - Bf), s = A(Cd - Bf) + h(FC - BG) + Be^2, t = A(FC - BG). \tag{14}$$

Solving (13a) for any of the field variables, say $u_z$, we have

$$u_z = \sum_{j=1}^{4} u_{zj} = \sum_{j=1}^{4} (A_j Z_0(v_j r) + B_j X_0(v_j r)) \tag{15}$$

in which

$$\nu_j = |\lambda_j^2|^{1/2},$$
$$Z_0(\nu_j r) = J_0(\nu_j r) \quad \text{or} \quad I_0(\nu_j r) \quad \text{according as} \quad \lambda_j^2 > 0 \quad \text{or} \quad <0,$$
$$X_0(\nu_j r) = I_0(\nu_j r) \quad \text{or} \quad K_0(\nu_j r) \quad \text{according as} \quad \lambda_j^2 > 0 \quad \text{or} \quad <0,$$

$J_0$, $Y_0$, and $I_0$, $K_0$ being Bessel functions and modified Bessel functions of the first and second kinds, all of order zero;

$(A_j, B_j)(j = 1, 2, 3, 4)$ denote eight arbitrary constants.

The set of eqns (13) indicates that the field parameters $u_z$, $\varphi$, $\psi$ and $v$ are linearly dependent on one another, so that one can write

$$v = \sum_{j=1}^{4} d_j u_{zj}, \; \psi = \sum_{j=1}^{4} b_j u_{zj} \quad \text{and} \quad \varphi = \sum_{j=1}^{4} C_j u_{zj}. \tag{16}$$

The constants $b_j$, $C_j$, $d_j (j = 1, 2, 3, 4)$ are evaluated if the expressions (16) are substituted into the set of differential equations taking (15) into consideration. We thus have

$$d_j = -\frac{l\lambda_j^4 - m\lambda_j^2 + n}{L\lambda_j^4 - M\lambda_j^2 + N}, \quad b_j = \frac{ed_j}{h\lambda_j^2 - A} \tag{17}$$

and

$$C_j = -\frac{1}{B\lambda_j^2}[(b\lambda_j - D) + (d\lambda_j^2 - F)d_j].$$

In order to incorporate the effect of the material damping of the bone specimen, the coefficients $C_{mn}$ in the constitutive relations (2) are to be regarded as functions of the time-derivative $D \equiv \partial/\partial t$. Bulanowski and Yeh[16] have shown that in the case of harmonic waves propagating through a viscoelastic continuum, the elastic moduli ($C_{mn}$) in the elastic solutions are to be replaced by the complex functions $C_{mn}(iw)$, $(i = \sqrt{-1})$.

## 3. WAVES PROPAGATING THROUGH A TUBULAR BONE

Let us now restrict our attention to a long tubular bone having $r_1$ and $r_2$ as the endosteal and periosteal radii respectively. We define the interface of two-phase material by $r = R$. For the purpose of discussing the propagation of axi-symmetric waves through the bone specimen, the tubular surfaces being assumed to be traction-free and to be maintained at zero potential, we can write

$$T_{rr} = T_{rz} = T_{r\theta} = v = 0 \text{ on } r = r_1, r_2. \tag{18}$$

Further, in conformity with the assumption of the continuity of the displacement and stress-components, the electric potential as also the electric displacement, one can write

$$\begin{aligned} &u_r^{(1)} = u_r^{(2)}, u_\theta^{(1)} = u_\theta^{(2)}, u_z^{(1)} = u_z^{(2)}, \\ &T_{rr}^{(1)} = T_{rr}^{(2)}, T_{r\theta}^{(1)} = T_{r\theta}^{(2)}, T_{rz}^{(1)} = T_{rz}^{(2)} \\ &v^{(1)} = v^{(2)}, D_r^{(1)} = D_r^{(2)} \quad \text{on } r = R \end{aligned} \tag{19}$$

(see [9, 14]) in which the superscripts (1) and (2) refer to quantities in the two different layers of the tubular bone specimen under consideration.

By making use of (2)–(4) together with (15)–(17) in (18) and (19) one can obtain a set of sixteen linear algebraic equations involving sixteen unknowns. Eliminating these unknown quantities, one can obtain the dispersion relation in the form of a determinant of order 16, equated to zero; the elements of the determinant, $D_{lk}$ are shown in Appendix. By solving this equation one can determine the wave propagation constant, therefrom the wave speed as also the attenuation of the waves due to the damping material behaviour of osseous tissues.

If bone piezoelectricity be ignored, the eighth order partial differential equations (13) reduce to the fourth order ones given by

$$(\nabla^2 + \lambda_1^2)(\nabla^2 + \lambda_2^2)(\varphi, u_z) = 0 \tag{20}$$

with

$$\lambda_1^2, \lambda_2^2 = \frac{1}{2ab}[(Ab + Da - B^2) \pm \{(Ab + Da - B^2)^2 - 4abAD\}^{1/2}]. \tag{21}$$

The solution of these equations can be written as

$$u_z = \sum_{j=1,2} u_{zj} = \sum_{j=1,2} (A_j Z_0(\nu_j r) + B_j X_0(\nu_j r), \quad \text{and} \quad \varphi = \sum_{j=1,2} G_j u_{zj}, j = 1, 2 \tag{22}$$

where the symbols $\nu_j$, $Z_0$, and $X_0$ retain their earlier definitions and

$$G_j = -ike_j, \quad \text{with} \quad e_j = \frac{C_{13} + C_{44}}{a\lambda_j^2 - A}. \tag{23}$$

In this case, the dispersion relation for a compact bone is given by

$$\|d_{nm}\| = 0 \tag{24}$$

where $d_{nm}$, the elements of a determinant of the fourth order, are given as

$$d_{nm} = \mu_m^2 E_m W''(\mu_m r) + \frac{C_{12}}{C_{11}} \frac{E_m}{r} W_0'(\mu_m r) + \frac{C_{13}}{C_{11}} W_0(\mu_m r),$$
$$(n = 1, 3; m = 1, 2, 3, 4)$$

(in this case $n = 1$ refers to $r = r_1$ and $n = 3$ to $r = r_2$)

$$d_{nm} = \mu_m W_0'(\mu_m r)[1 - k^2 E_m], (n = 2, 4, m = 1, 2, 3, 4)$$

(here $n = 2, 4$ refer respectively to $r = r_1, r_2$) with

$$E_1 = E_3 = e_1, E_2 = E_4 = e_2; \quad \mu_1 = \mu_3 = \nu_1, \mu_1 = \mu_3 = \nu_2,$$

and

$$m = 1, 2 \quad \text{refer to} \quad W_0 = Z_0 \quad \text{and} \quad m = 3, 4 \text{ to } W_0 = X_0. \tag{25}$$

The prime over a function denotes differentiation with respect to its argument.

## 4. NUMERICAL RESULTS AND CONCLUSIONS

Due to non-availability of all the requisite experimental data necessary for characterizing the two-phase material behaviour of bones, the computational work has been rest.icted to a compact (single-layered) bone with use of following data (see [9, 17])

$$\rho = 2000\,\text{kg/m}^3,\ C_{11} = 2.38,\ C_{33} = 3.34,\ C_{13} = 1.2,\ C_{12} = 1.02$$
$$C_{66} = .68, \text{all in } 10^{10}\,\text{N/m}^2;\quad r_1 = .0038\,\text{m},\ r_2/r_1 = 1.7.$$

Lakes *et al.* [6], on the basis of their experimental observations remarked that the elastic moduli and the loss tangents could be considered almost frequency-independent at the ultrasound range; so in this range, the viscoelastic effects may be incorporated on replacing the elastic constants $C_{mn}$ by $C_{mn}(1 + i\delta)$; $\delta$ being the loss tangent. For $w = 2\pi \times 10^5$ rads/sec., the value of $\delta$ approximately equals to $\delta = 0.01$ [6]. Keeping in mind that for a viscoelastic material, $k$ is a complex quantity, the real part being the representative of the wave propagation constant and imaginary part that of the attenuation coefficient, and using the above mentioned values of $w$ and $\delta$, the dispersion relation is programmed on a high speed digital computer for determining $k$. The dispersion equation being transcendental will possess an infinite number of roots. The first few values of the computed roots are presented in Table 1. For the purpose of comparison, the results presented by Vayo and Ghista [9] are also shown in the table. They did not account for the bonepiezoelectricity as well as the energy dissipation in their analysis which was carried out by an entirely different procedure. Further, their results are restricted to "very short" wave length (asymptotic expressions being used for the Bessel functions) and as a result the traction-free conditions of the periosteal surface of the long bone specimen could not be made use of. It may be noted that the values of the wave propagation constant obtained by us for the first and second modes are closed to the value obtained by Vayo and Ghista [9] for the first mode, while our computed values for the third and fourth modes correspond to the value obtained by them for the second mode. This observation may be attributed to the occurrence of satellite modes in the vicinity of principal modes. The so called satellite modes can be detected only when, for an analytical study, the analysis is performed by taking into account the finer aspects (both physical and mathematical) of the problem.

It has been remarked in Sapriel [13] that for a plane wave propagation through a piezoelectric medium, the wave characteristics can be estimated from the corresponding solution valid for a non-piezoelectric medium if the elastic moduli $C_{mn}$ are replaced by their modulated values, $C'_{mn}$. For the plane axial waves, following Sapriel [13], we can write

$$C'_{mn} = C_{mn} + \frac{e_{3m}e_{3n}}{\epsilon_{33}}. \tag{26}$$

Table 1. Values of the wave propagation constants and attenuation coefficients at different modes

| Mode No. : (j) | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| Wave propagation constant ($k_j$ in $m^{-1}$) (Vayo and Ghista, 1971) | 186 | 352 | 442 | | |
| Wave propagation constant ($k_j$ in $m^{-1}$) (Present analysis) | 145 | 155 | 312.5 | 315.5 | 416.5 |
| Attenuation coefficient, ($\gamma_j$ in $m^{-1}$) | 0.79 | 0.698 | 0.169 | 0.167 | 0.103 |

In that case, the amplitude of the electric potential $v_0$ induced by the travelling waves, is proportional to the amplitude of the axial displacement $u_z^0$ so that

$$v_0 = e_{33} u_z^0. \tag{27}$$

For a bone medium, $e_{31} = e_{32} = 63.4 \times 10^{-5}\ \mathrm{C/m^2}$, $e_{33} = 54.8 \times 10^{-5}\ \mathrm{C/m^2}$ and $\epsilon_{33} = 1.33 \times 10^{-9}\ \mathrm{C/vm}$ (*cf* Güzelsu[12]). Then the correction term in (26) is of the order of $10^4\ \mathrm{N/m^2}$, whereas the elastic moduli are of the order of $10^{10}\ \mathrm{N/m^2}$.

In a situation as this, $C'_{mn} \simeq C_{mn}$. This asserts that for plane wave propagation, the wave characteristics for a piezoelectric medium are almost identical to those for a corresponding non-piezoelectric medium. Thus for points in a long bone specimen, which are at moderate distances from the origin of waves and the surfaces of discontinuity, our results in Table 1, may be considered as nearly equal to those obtained by incorporating bone piezoelectricity, the induced elastic field being determined by (27).

## 5. REMARKS

In a recent study of Saha and Güzelsu[18], the anti-symmetric electromechanical wave propagation was considered by treating bone as a hexagonal polar material exhibiting piezoelectric effects; however, the experimentally established dissipative material behaviour of osseous tissues was not paid due attention by them. In solving the problem of wave propagation through a hollow piezoelectric cylinder as the representative of a tubular bone specimen, the authors have ignored the piezoelectric coefficients in the constitutive equations. This led the derivation of the dispersion equation considerably simple. Such dispersion equations had already been analytically obtained and numerically solved by Mirsky[19], but Saha and Güzelsu[18] have numerically solved this dispersion relation for a particular bone specimen to obtain useful results for the purpose of correlating them with the findings of certain experiments performed with similar bone specimens. By solving the electromagnetic boundary conditions, the values of the external magnetic field induced by the travelling antisymmetric wave have also been obtained. In fact, this was the principal aim of the authors in this analytical study. It has been claimed that their theoretical results closely agree with the experimental ones performed with magnetic sensors.

The purpose of our analysis is to incorporate the viscoelastic as well as the piezoelectric properties of osseous tissues in the problem considered by Vayo and Ghista[9]. As reported by Chan *et al.*[20], the ultrasound wave propagating through a viscoelastic bone medium can result in a temperature rise (a knowledge of which is useful in obtaining criteria for dosage) which can be measured if we have an idea of the wave attenuation coefficients. A consideration of bone piezoelectricity in the constitutive relations makes the dispersion relation very much complicated, even for the simple axisymmetric waves that are considered in this paper. This is due to the fact that even for an axisymmetric analysis of the problem of wave propagation through a piezoelectric cylindrical shell one cannot assume the circumferential displacement, $u_\theta$ to be zero (see[12]), which is exactly the case in[18] with $n = 0$. Because of this difference in analytical approach one cannot compare the results of the present study with those of Saha and Güzelsu[18], though an axisymmetric motion may be taken as a particular case of general antisymmetric motion.

The principal aim of providing the computational results for a simple case, in the present analysis, is to demonstrate that under a similar situation it gives the results which are in close agreement with those in[9]. From the present study, one may conclude that the viscoelastic and piezoelectric effects in bone media can be looked upon as small perturbation effects to a general elastic material behaviour. Numerical solution of the general dispersion relation by considering all such perturbation effects may certainly be obtained. But it has already been conjectured that it would only increase the number of the so called 'satellites'.

The magnetic field induced by a travelling torsional wave in tubular bones was recently discussed by the present authors[8].

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## APPENDIX

The elements of the 16th order determinant $D_{lk}$ are defined below.

$$D_{lk} = C_{11}^{(j)} C_m^{(j)} W_0''(\nu_m^{(j)} r) + \frac{C_{12}^{(j)} C_m^{(j)} W_0'(\nu_m^{(j)} r)}{r} - ik(C_{13}^{(j)} + e_{31}^{(j)} d_m^{(j)}) W_0(\nu_m^{(j)} r)$$

($l = 1, 5$; $k = 1, 2, 3, 4, 5, 6, 7, 8$; $l = 1$ refers to $j = 1$ and $r = r_1$; $l = 5$ to $j = 2$ and $r = r_2$)

$D_{lk} = 0$, $l = 1, 5$; $k = 9, 10, \ldots, 16$.

$D_{lk} = [C_{44}^{(j)}(1 - ikC_m^{(j)}) + e_{15}^{(j)} d_m^{(j)}] W_0'(\nu_m^{(j)} r)$ ($l = 2, 6$; $k = 1, 2, \ldots 8$; $l = 2$ refers to $j = 1$, $r = r_1$ and $l = 6$ to $j = 2$, $r = r_2$)

$D_{lk} = 0$ ($l = 2, 6$; $k = 9, 10, \ldots 16$)

$$D_{lk} = C_{66}^{(j)} b_m^{(j)} \left[ W_0''(\nu_m^{(j)} r) - \frac{W_0'(\nu_n^{(j)} r)}{r} \right]$$ ($l = 3, 7$; $k = 1, 2, \ldots 8$;

$l = 3$ refers to $j = 1$, $r = r_1$ and $l = 7$ to $j = 2$, $r = r_2$)

$D_{lk} = 0$($l = 3, 7$; $k = 9, 10, \ldots 16$).

$D_{lk} = d_m^{(j)} W_0(\nu_m^{(j)} r)$($l = 4, 8$, $k = 1, 2, \ldots 8$; $l = 4$ refers to $j = 1$, $r = r_1$ and $l = 7$ to $j = 2$, $r = r_2$)

$D_{lk} = 0$ ($l = 4, 8$; $k = 9, 10, \ldots 16$).

$$D_{lk} = C_{11}^{(1)} C_m^{(1)} W_0''(\nu_m^{(1)} R) + \frac{C_{12}^{(1)} C_m^{(1)} W_0'(\nu_m^{(1)} R)}{R} - ik(C_{13}^{(1)} + e_{31}^{(1)} d_m^{(1)}) W_0(\nu_m^{(1)} R) \quad (l = 9, k = 1, 2, 3, \ldots 8)$$

$$D_{lk} = -\left[ C_{11}^{(2)} C_m^{(2)} W_0''(\nu_m^{(2)} R) + \frac{C_{12}^{(2)} C_m^{(2)} W_0'(\nu_m^{(2)} R)}{2} - ik(C_{13}^{(2)} + e_{31}^{(2)} d_m^{(2)}) W_0(\nu_m^{(2)} R) \right] (l = 9, k = 9, 10 \ldots 16)$$

$D_{lk} = [C_{44}^{(1)}(1 - ikC_m^{(1)}) + e_{15}^{(1)} d_m^{(1)}] W_0'(\nu_m^{(1)} R)$, ($l = 10$, $k = 1, 2, \ldots 8$)

$D_{lk} = -[C_{44}^{(2)}(1 - ikC_m^{(2)}) + e_{15}^{(2)} d_m^{(2)}] W_0'(\nu_m^{(2)} R)$, ($l = 10$, $k = 9, 10, \ldots 16$)

$D_{lk} = -C_{66}^{(2)} b_m^{(2)}(W_0''(\nu_m^{(2)} R) - W_0'(\nu_m^{(1)} R)/R]$, ($l = 11$, $k = 1, 2, \ldots 8$)

$D_{lk} = -C_{66}^{(2)} b_b^{(2)}(W_0''(\nu_m^{(2)} R) - W_0'(\nu_m^{(2)} R)/R]$, ($l = 11$, $k = 9, 10, \ldots 16$)

$D_{lk} = d_m^{(1)} W_0(\nu_m^{(1)} R)$, ($l = 12$, $k = 1, 2, \ldots 8$)

$D_{lk} = -d_m^{(2)} W_0(\nu_m^{(2)} R)$, ($l = 12$, $k = 9, 10, \ldots 16$).

$D_{lk} = [-ike_{14}^{(1)} b_m^{(1)} + e_{15}^{(1)} - ike_{15}^{(1)} C_m^{(1)} + \epsilon_{11}^{(1)} d_m^{(1)}] W_0'(\nu_m^{(1)} R)$, ($l = 13$, $k = 1, 2, \ldots 8$)

$D_{lk} = -[-ike_{14}^{(2)} b_m^{(2)} + e_{15}^{(2)} - ike_{15}^{(2)} C_m^{(2)} + \epsilon_{11}^{(2)} d_m^{(2)}] W_0'(\nu_m^{(2)} R)$ ($l = 13$, $k = 9, \ldots 16$)

$D_{lk} = W_0(\nu_m^{(1)} R)$, ($l = 14$, $k = 1, 2, \ldots 8$)

$D_{lk} = W_0(\nu_m^{(2)} R)$($l = 14$, $k = 9, \ldots 16$)

$D_{lk} = C_m^{(1)} W_0(\nu_m^{(1)} R)$($l = 15$, $k = 1, 2, \ldots 8$)

$D_{lk} = -C_m^{(2)} W_0'(\nu_m^{(2)} R)$($l = 15$, $k = 9, \ldots 16$)

$D_{lk} = b_m^{(1)} W_0'(\nu_m^{(1)} R)$($l = 16$, $k = 1, \ldots 8$)

$D_{lk} = -b_m^{(2)} W_0(\nu_m^{(2)} R)$($l = 16$, $k = 9, \ldots 16$).

The prime denotes differentiation with respect to $r$; and in all the above cases

$k = 1, 5, 9, 13$ refer to $m = 1$, $k = 2, 6, 10, 14$ to $m = 2$, $k = 3, 7, 11, 15$ to $m = 3$, $k = 4, 8, 12, 16$ to $m = 4$; $k = 1, 2, 3, 4, 9, 10, 11, 12$ refer to $W_0 = Z_0$, and $k = 5, 6, 7, 8, 13, 14, 15, 16$ to $W_0 = X_0$.